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EVOLUTION EQUATION FOR WEAKLY NONLINEAR WAVES IN A TWO-LAYER FLUID WITH GENTLY SLOPING BOTTOM AND LID

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A second-order differential model for three-dimensional perturbations of the interface of two fluids of different density is constructed. An evolution equation for traveling quasistationary waves of arbitrary length and small but finite amplitude is obtained. In the case of the horizontal bottom and lid, there are perturbations of the Stokes-wave type among steady-state periodic solutions. For moderately long perturbations, solutions in the form of solitary waves which are in agreement with the available experimental and analytical results are found. The problem of a smooth transition from the deep-fluid to the shallow-fluid region is studied.

Borisov and Khabakhpashev proposed a very simple differential model capable of describing the dynamics of long and short three-dimensional, weakly nonlinear perturbations of the interface of two fluids of different density confined by a rigid horizontal bottom and lid. However, the derivation of the wave-type equation for quasistationary disturbances was not quite correct. In addition, formally, even a linearized equation can have unstable solutions. The purpose of this work is to obtain a second-order differential model and a corresponding evolution equation that is free from the above-mentioned disadvantages without requiring the layers to be of constant depth.

1. Second-Order Differential Model. It is assumed that the fluids are ideal, incompressible, and immiscible, the stationary components of the fluid motion equal zero, the occurring oscillating flows are potential, and the waves are weakly nonlinear (i.e., $\eta_a k/\tanh(kh_m) \sim \varepsilon$, where η_a is the amplitude of the disturbance at the interface, k is the wave number, h_m is the depth of the smaller layer, and ε is a small parameter). Third-order infinitesimals are omitted with capillary effects ignored.

In [1], the initial system of hydrodynamic equation was reduced to the equations

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \{\langle \boldsymbol{u}_l \rangle [\eta + (-1)^l h_l]\} = 0, \tag{1.1}$$

$$\frac{\partial \boldsymbol{u}_{li}}{\partial t} + \nabla\Big(g\eta + \frac{u_{li}^2}{2} + \frac{p_i}{\rho_l}\Big) + \frac{\partial^2 \eta}{\partial t^2}\nabla\eta = 0 \tag{1.2}$$

by integrating over the vertical coordinate and by using the standard kinematic and dynamic boundary conditions on the lid, the bottom, and the interface. Here t is the time, $\boldsymbol{u}$ is the vector of the horizontal component of the fluid velocity, the angular brackets indicate its value averaged over the layer depth, g is the acceleration of gravity, ρ is the density, p is the pressure, $l = 1$ for the upper fluid, and $l = 2$ for the lower one; the subscript i indicates the values of the quantities related to the interface and the gradient operator ∇ is determined in the horizontal plane.

Then, in [1], the use of well-known dependences for the vertical profiles $\boldsymbol{u}_l$ (see, for example, [2]) enabled one to relate the Fourier components of the boundary and averaged velocities of the fluids:

$$u_{li}(\omega, k) = kh_l \coth(kh_l)\langle u_l(\omega, k)\rangle, \tag{1.3}$$

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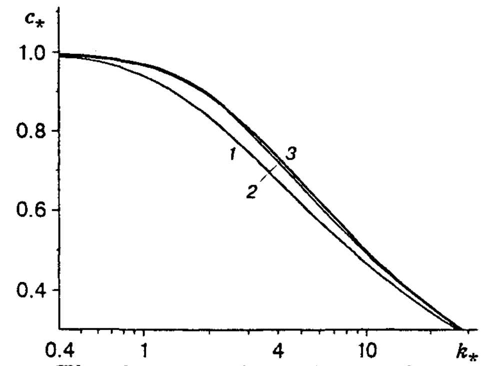

Fig. 1. The dimensionless phase velocity c_* as a function of the dimensionless wave number k_* for $h_2/h_1 = 3$ and $\rho_2/\rho_1 = 1.25$: curves 1 and 2 refer to calculation by the approximation (1.5) for $\alpha = 0$ and $2/3$, respectively, and curve 3 to calculation by the exact dispersion relation (1.4).

where ω is the cyclic frequency. If $\nabla h \sim \varepsilon^{3/2}$, the formulas for waves in a liquid of constant depth are also locally true for a weakly inclined bottom and lid. In this connection, we give one more classical formula, namely, the dispersion relation for linear monoharmonic vortex-free waves in a two-layer liquid [2]:

$$\omega^2[\rho_1 \coth(kh_1) + \rho_2 \coth(kh_2)] = gk(\rho_2 - \rho_1). \tag{1.4}$$

We replace approximately the transcendental equation (for the wave number) by the following simplest Padé approximation:

$$\omega^2(1/A_\omega + \omega_*^2) = \omega^2/c_*^2 = k^2c_0^2 = k^2 g\delta, \quad A_\omega = 1 + \alpha\omega_*^2, \quad \omega_*^2 = \omega^2\beta/g_+^-,$$
$$g_+^- = g\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}, \qquad \beta = h_1h_2\frac{\rho_1 + \rho_2}{\chi}, \qquad \delta = h_1h_2\frac{\rho_2 - \rho_1}{\chi}, \qquad \chi = \rho_1h_2 + \rho_2h_1. \tag{1.5}$$

Here c is the phase velocity (the subscript 0 indicates its value for waves of infinitely small frequency) and α is the numerical coefficient. If $\alpha = 0$, we have the simplest polynomial approximation suggested in [1]. In this case, the long-wave ($\omega_*^2 \ll 1$) and short-wave ($\omega_*^2 \gg 1$) asymptotic curves coincide with the exact dispersion curve. In addition, in the range of intermediate frequencies the approximation error is determined by the ratios of the layer depths and the fluid densities. In particular, for $h_2/h_1 = 3$ and $\rho_2/\rho_1 = 1.25$ (the experiment in [3] was performed for these values), the maximum relative deviation of the exact relation (1.4) from the approximate (1.5) with $\alpha = 0$ is 8.5%, and it is reached for $k_* = kH \approx 4$ ($H = h_1 + h_2$ is the distance between the bottom and the lid). If $\alpha = 2/3$, the approximation error does not exceed 2%, and the corresponding maximum is attained for $k_* \approx 6$ (Fig. 1). A comparison of relations (1.4) and (1.5) leads to the expressions $kh_l \coth(kh_l) = 1/A_\omega + \omega^2 h_l/g_+^-$ by means of which we write Eqs. (1.3) in the form

$$A_\omega u_{li}(\omega, k) = (1 + A_\omega\omega^2 h_l/g_+^-)\langle u_l(\omega, k)\rangle. \tag{1.6}$$

Application of the inverse Fourier transform to (1.6) yields the differential relations for the boundary and averaged velocities

$$A_t \boldsymbol{u}_{li} = \langle \boldsymbol{u}_l\rangle - A_t\frac{h_l}{g_+^-}\frac{\partial^2\langle \boldsymbol{u}_l\rangle}{\partial t^2}, \qquad A_t = 1 - \alpha\frac{\beta}{g_+^-}\frac{\partial^2}{\partial t^2}. \tag{1.7}$$

In contrast to formulas (1.7), relations (1.3) and (1.6) also remain valid in the case of weakly nonlinear perturbations. Therefore, we can use the generalization of expressions (1.7)

$$A_t\boldsymbol{u}_{li} = \langle \boldsymbol{u}_l\rangle - A_t\Big\{\Big[\frac{h_l}{g_+^-} + \frac{(-1)^l}{g_+^-}\Big(\eta + \frac{3h_l}{g_+^-}\frac{\partial^2\eta}{\partial t^2}\Big)\Big]\frac{\partial^2\langle \boldsymbol{u}_l\rangle}{\partial t^2} - 4\boldsymbol{u}_{li}'\Big\}, \tag{1.8}$$

where $\boldsymbol{u}'_{li}$ are the velocities of translational motion of the fluid particles near the interface of the infinitely deep layers in the direction of the wave motion. These values are the second-order infinitesimals and depend neither on the horizontal coordinates nor on time. Thus, in the approximation considered (without including the third-order infinitesimals) the terms $\partial \boldsymbol{u}_{li}/\partial t$ and ∇u_{li}^2, which enter Eqs. (1.2), does not contain $\boldsymbol{u}'_{li}$. However, the presence of $\boldsymbol{u}'_{li}$ in Eqs. (1.8) is necessary for obtaining a solution of the Stokes-wave type for two infinitely deep layers (see, e.g., [4, 5]). We note that relations (1.8) are also in agreement with the results available for the long-wave range.

We now differentiate Eqs. (1.2) twice with respect to time, multiply all the terms by $-\alpha\beta/g_+^-$, and add again to Eqs. (1.2):

$$A_t \frac{\partial \boldsymbol{u}_{li}}{\partial t} + A_t\Big[\nabla\Big(g\eta + \frac{p_i}{\rho_l}\Big) + \frac{\partial^2 \eta}{\partial t^2}\nabla\eta\Big] + \frac{1}{2}A_t \nabla u_{li}^2 = 0. \tag{1.9}$$

The first terms in Eqs. (1.9) are replaced by means of expressions (1.8). Owing to the nonlinearity of the last terms of Eqs. (1.9), in the substitution one can confine oneself to the first order of accuracy and the third-order infinitesimals can be ignored, i.e., one can assume that

$$\boldsymbol{u}_{li} = \langle \boldsymbol{u}_l\rangle - (h_l/g_+^-)(\partial^2\langle \boldsymbol{u}_l\rangle/\partial t^2).$$

Therefore, the conservation laws for the horizontal components of the momentum in the layers do not contain the values of the fluid velocities at the interface:

$$\frac{\partial\langle \boldsymbol{u}_l\rangle}{\partial t} + A_t\Big\{\nabla\Big[g\eta + \frac{1}{2}\Big(\langle u_l\rangle - \frac{h_l}{g_+^-}\frac{\partial^2\langle u_l\rangle}{\partial t^2}\Big)^2 + \frac{p_i}{\rho_l}\Big] + \frac{\partial^2\eta}{\partial t^2}\nabla\eta\Big\}$$
$$-A_t\frac{\partial}{\partial t}\Big\{\Big[\frac{h_l}{g_+^-} + \frac{(-1)^l}{g_+^-}\Big(\eta + \frac{3h_l}{g_+^-}\frac{\partial^2\eta}{\partial t^2}\Big)\Big]\frac{\partial^2\langle \boldsymbol{u}_l\rangle}{\partial t^2}\Big\} = 0. \tag{1.10}$$

To reduce the system of four differential equations (1.1) and (1.10) to one equation for the perturbation of the interface, we apply the operator ∇ to Eqs. (1.10) in a scalar manner and multiply them by $(-1)^{l+1}h_l$:

$$\Big(1 - \frac{h_l}{g_+^-}A_t\frac{\partial^2}{\partial t^2}\Big)\frac{\partial}{\partial t}[(-1)^{l+1}h_l\nabla\cdot\langle \boldsymbol{u}_l\rangle]$$
$$-(-1)^l A_t\Big[g h_l\nabla^2\eta + \frac{h_l}{\rho_l}\nabla^2 p_i + h_l\nabla\cdot\Big(\frac{\partial^2\eta}{\partial t^2}\nabla\eta\Big) + \frac{h_l}{2}\nabla^2\Big(\langle u_l\rangle - \frac{h_l}{g_+^-}\frac{\partial^2\langle u_l\rangle}{\partial t^2}\Big)^2\Big]$$
$$+\frac{h_l}{g_+^-}\nabla\cdot\Big\{A_t\frac{\partial}{\partial t}\Big[\Big(\eta + \frac{3h_l}{g_+^-}\frac{\partial^2\eta}{\partial t^2}\Big)\frac{\partial^2\langle \boldsymbol{u}_l\rangle}{\partial t^2}\Big]\Big\} - (-1)^l A_t\frac{h_l}{g_+^-}\frac{\partial^3\langle \boldsymbol{u}_l\rangle}{\partial t^3}\cdot\nabla h_l$$
$$+(-1)^l\frac{\alpha\beta h_l}{g_+^- h_1 h_2\chi}\frac{\partial^2}{\partial t^2}\Big(g\nabla\eta + \frac{\nabla p_i}{\rho_l} - \frac{h_l}{g_+^-}\frac{\partial^3\langle \boldsymbol{u}_l\rangle}{\partial t^3}\Big)\cdot(\rho_1 h_2^2\nabla h_1 + \rho_2 h_1^2\nabla h_2) = 0. \tag{1.11}$$

The mass conservation laws (1.1) are more conveniently rewritten in the form

$$(-1)^{l+1}h_l\nabla\cdot\langle \boldsymbol{u}_l\rangle = \frac{\partial\eta}{\partial t} + \eta\nabla\cdot\langle \boldsymbol{u}_l\rangle + \langle \boldsymbol{u}_l\rangle\cdot\nabla[\eta + (-1)^l h_l].$$

We substitute these relations into the first linear terms of Eqs. (1.11):

$$\Big(1 - \frac{h_l}{g_+^-}A_t\frac{\partial^2}{\partial t^2}\Big)\frac{\partial}{\partial t}\Big[\frac{\partial\eta}{\partial t} + \nabla\cdot(\langle \boldsymbol{u}_l\rangle\eta)\Big] + \frac{h_l}{g_+^-}A_t\frac{\partial}{\partial t}\nabla\cdot\Big[\Big(\eta + \frac{3h_l}{g_+^-}\frac{\partial^2\eta}{\partial t^2}\Big)\frac{\partial^2\langle \boldsymbol{u}_l\rangle}{\partial t^2}\Big]$$
$$-(-1)^l\Big\{A_t\Big[g h_l\nabla^2\eta + \frac{h_l}{\rho_l}\nabla^2 p_i + h_l\nabla\cdot\Big(\frac{\partial^2\eta}{\partial t^2}\nabla\eta\Big) + \frac{h_l}{2}\nabla^2\Big(\langle u_l\rangle - \frac{h_l}{g_+^-}\frac{\partial^2\langle u_l\rangle}{\partial t^2}\Big)^2\Big] + \frac{\partial\langle \boldsymbol{u}_l\rangle}{\partial t}\cdot\nabla h_l\Big\}$$
$$+(-1)^l\frac{\alpha\beta h_l}{g_+^- h_1 h_2\chi}\frac{\partial^2}{\partial t^2}\Big(g\nabla\eta + \frac{\nabla p_i}{\rho_l} - \frac{h_l}{g_+^-}\frac{\partial^3\langle \boldsymbol{u}_l\rangle}{\partial t^3}\Big)\cdot(\rho_1 h_2^2\nabla h_1 + \rho_2 h_1^2\nabla h_2) = 0. \tag{1.12}$$

In Sec. 4, these equations are used for the solution of the problem of a smooth transition of the linear wave from a deep to a shallow fluid.

2. Evolution Equation for Quasistationary Waves. In Eqs. (1.12), the averaged velocities of the fluids enter only into the second-order infinitesimal terms. To eliminate $\langle \boldsymbol{u}_l \rangle$ from these terms, it is necessary to make an additional assumption. We assume that in the reference system moving together with the wave the form of perturbation varies slowly, i.e., $\eta = \eta(\tau, \boldsymbol{r})$, where $\tau = \varepsilon t$ and $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{U}t$ [$\boldsymbol{x} = (x, y)$ and $\boldsymbol{U}$ is the characteristic wave velocity]. Then, we have $\nabla_{\boldsymbol{x}} = \nabla_{\boldsymbol{r}}$ and $\partial/\partial t = \varepsilon \partial/\partial\tau - D$, where $D = \boldsymbol{U} \cdot \nabla$. Hence, with accuracy up to the first-order infinitesimal terms from Eqs. (1.1) we have the equalities $\langle \boldsymbol{u}_l \rangle = (-1)^l \boldsymbol{U}\eta/h_l$ by means of which we substitute $\langle \boldsymbol{u}_l \rangle$ into Eqs. (1.12):

$$D^2\eta - \frac{h_l}{g_+^-} A_D D^4\eta - \frac{(-1)^l}{h_l} D^2\eta^2 - (-1)^l A_D \Big[g h_l \nabla^2\eta + \frac{h_l}{\rho_l}\nabla^2 p_i + \frac{U^2}{2h_l}\nabla^2\eta^2 + h_l \nabla\cdot(D^2\eta\nabla\eta)\Big]$$

$$+\frac{(-1)^l}{g_+^-} A_D \Big[D^4\eta^2 + (D^2 - U^2\nabla^2)(\eta D^2\eta) - \frac{h_l}{2g_+^-}(6D^2 + U^2\nabla^2)(D^2\eta)^2\Big] - \frac{1}{h_l} D\eta D h_l$$

$$+\frac{\alpha\beta D^2}{g_+^- h_1 h_2 \chi}\Big[(-1)^l\Big(g h_l \nabla\eta + \frac{h_l}{\rho_l}\nabla p_i\Big) - \frac{h_l}{g_+^-} D^3\eta \boldsymbol{U}\Big] \cdot (\rho_1 h_2^2 \nabla h_1 + \rho_2 h_1^2 \nabla h_2)$$

$$= 2\varepsilon\Big[1 - \frac{2h_l}{g_+^-}D^2 + \alpha\frac{h_l\beta}{g_+^-}\Big(g\nabla^2 + \frac{3}{g_+^-}D^4\Big)\Big] D\frac{\partial\eta}{\partial\tau}. \qquad (2.1)$$

Here $A_D = 1 - \alpha\beta D^2/g_+^-$, and the third-order infinitesimal terms are omitted as in the previous transformations.

We obtained a system of two equations for two desired η and p_i. To reduce them to one equation, we multiply Eq. (2.1) by h_2/ρ_2 for $l = 1$ and h_1/ρ_1 for $l = 2$ and add them. As a result, we obtain the evolution equation

$$D^2\eta - \zeta D^2\eta^2 - A_D\Big[g\delta\nabla^2\eta + \frac{\beta}{g_+^-}A_D D^4\eta + \zeta\frac{U^2}{2}\nabla^2\eta^2 + \delta\nabla\cdot(D^2\eta\nabla\eta)\Big]$$

$$+A_D\Big\{\frac{\gamma\beta}{g_+^-}[D^4\eta^2 - (D^2 - U^2\nabla^2)(\eta D^2\eta)] - \frac{\delta}{2(g_+^-)^2}(6D^2 + U^2\nabla^2)(D^2\eta)^2\Big\}$$

$$-\Big[\frac{\boldsymbol{U}D\eta}{h_1 h_2\chi} - \frac{\alpha\beta(\rho_1+\rho_2)}{g_+^-\chi^2}D^2\Big(g_+^-\nabla\eta + \frac{\boldsymbol{U}}{g_+^-}D^3\eta\Big)\Big] \cdot (\rho_1 h_2^2\nabla h_1 + \rho_2 h_1^2 \nabla h_2)$$

$$= 2\varepsilon\Big[1 - \frac{2\beta}{g_+^-}D^2 + \alpha\frac{\beta^2}{g_+^-}\Big(g\nabla^2 + \frac{3}{g_+^-}D^4\Big)\Big] D\frac{\partial\eta}{\partial\tau} \qquad (2.2)$$

for perturbations of the interface of a two-layer fluid. In Eq. (2.2), all the coefficients are determined only by the physical (ρ_1, ρ_2, and g) and geometrical (h_1 and h_2) parameters of the system:

$$\zeta = \frac{\rho_2 h_1^2 - \rho_1 h_2^2}{h_1 h_2 \chi}, \qquad \gamma = \frac{\rho_2 h_1 - \rho_1 h_2}{h_1 h_2(\rho_1 + \rho_2)}.$$

The expressions for the quantities β, δ, χ, and g_+^- are given after formula (1.5).

We shall consider the most interesting partial cases. The layer depths are assumed to be constant and the plane wave is assumed to travel in the direction of increase of the x-coordinate. Then, Eq. (2.2) is

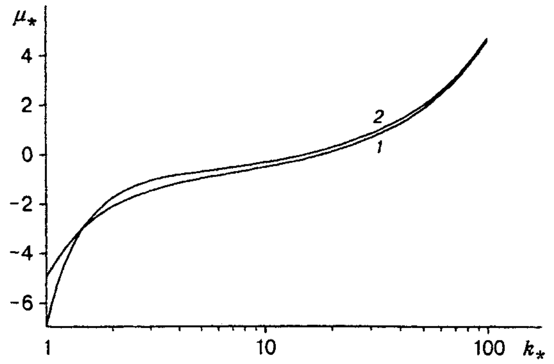

Fig. 2. Coefficient $\mu_* = \mu H$ as a function of the wavenumber k_* for $h_2/h_1 = 3$ and $\rho_2/\rho_1 = 1.25$: curves 1 and 2 refer to calculation by the approximation (2.6) for $\alpha = 0$ and $2/3$, respectively.

noticeably simplified:

$$2\varepsilon\left[1 - \frac{2\beta U^2}{g_+^-}\frac{\partial^2}{\partial\xi^2} + \frac{\alpha\beta^2}{g_+^-}\frac{\partial^2}{\partial\xi^2}\left(1 + \frac{3U^4}{g_+^-}\frac{\partial^2}{\partial\xi^2}\right)\right]\frac{\partial^2\eta}{\partial\tau\partial\xi} = (U^2 - g\delta A_\xi)\frac{\partial^2\eta}{\partial\xi^2} - \beta\frac{U^4}{g_+^-}A_\xi\frac{\partial^4\eta}{\partial\xi^4}$$

$$-\zeta U^2\left(1 + \frac{1}{2}A_\xi\right)\frac{\partial^2\eta^2}{\partial\xi^2} + \gamma\beta\frac{U^4}{g_+^-}A_\xi\frac{\partial^4\eta^2}{\partial\xi^4} - \frac{\delta}{2}U^2A_\xi\frac{\partial^2}{\partial\xi^2}\left[\left(\frac{\partial\eta}{\partial\xi}\right)^2 + \frac{7U^4}{(g_+^-)^2}\left(\frac{\partial^2\eta}{\partial\xi^2}\right)^2\right], \tag{2.3}$$

$$A_\xi = 1 - \alpha\beta\frac{U^2}{g_+^-}\frac{\partial^2}{\partial\xi^2}.$$

Here $\xi = x - Ut$. We emphasize that, for linear perturbations, this evolution equation gives only neutral-stable solutions. Equation (2.3) can be integrated over the variable ξ. In this case, the integration constant should be set equal to zero to eliminate nonphysical solutions. If a progressive stable wave occurs, we have $\partial\eta/\partial\tau = 0$, i.e., the left side of Eq. (2.3) is equal to zero. As a result, we obtain the standard differential equation

$$(U^2 - g\delta A_\xi)\frac{d\eta}{d\xi} - \beta\frac{U^4}{g_+^-}A_\xi\frac{d^3\eta}{d\xi^3} - \zeta U^2\left(1 + \frac{1}{2}A_\xi\right)\frac{d\eta^2}{d\xi}$$

$$+\gamma\beta\frac{U^4}{g_+^-}A_\xi\frac{d^3\eta^2}{d\xi^3} - \frac{\delta}{2}U^2A_\xi\frac{d}{d\xi}\left[\left(\frac{d\eta}{d\xi}\right)^2 + \frac{7U^4}{(g_+^-)^2}\left(\frac{d^2\eta}{d\xi^2}\right)^2\right] = 0. \tag{2.4}$$

One of the partial solutions of Eq. (2.4) is a solution of the Stokes-wave type, which can be written in the following form:

$$\eta = a\cos(k\xi) + \mu a^2\cos(2k\xi). \tag{2.5}$$

Substituting this relation into Eq. (2.4) and grouping the terms to the first power of a, we find that $U = c = \omega/k$. The terms that are quadratic in a lead to a more cumbersome Padé approximation of the coefficient μ:

$$\mu = \frac{3\zeta + \omega_*^2(8\gamma - \delta/\beta^2) + \omega_*^4 6\delta/\beta^2 + \alpha\omega_*^2[4\zeta + 32\gamma\omega_*^2 + 3\omega_*^2(9\omega_*^2 - 1/c_*^2)\delta/\beta^2]}{12\omega_*^2[1 + \alpha(5\omega_*^2 - 1/c_*^2)]}. \tag{2.6}$$

In the case where the fluids are very deep or the waves are moderately long (for $\alpha = 2/3$ and $h_1 = h_2$), this formula yields the results given in [4] (see also [5]).

The behavior of the coefficient μ in the range of medium and short waves is shown in Fig. 2. Three periods of wave profiles are shown in Fig. 3: in the case where the wavelength and the layer depths are of the same order of magnitude (Fig. 3a) and in the case where the wavelength is one order of magnitude lower than the layer depths (Fig. 3b). One can see that the hollows are steeper for negative values of the coefficient μ.

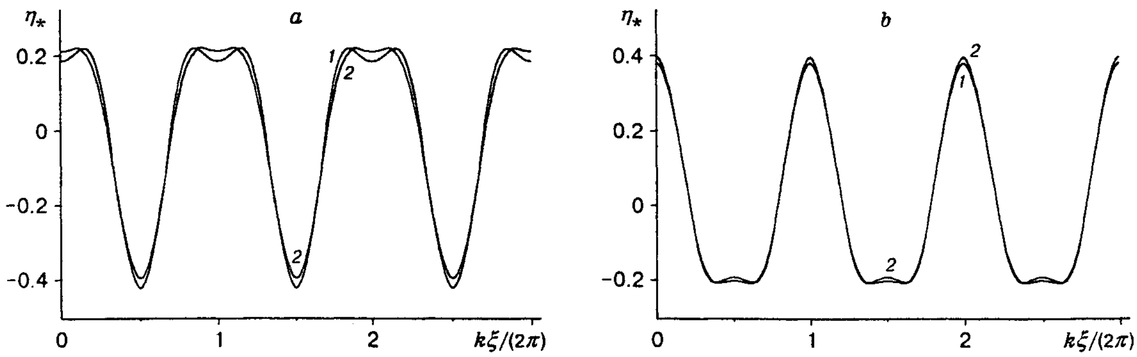

Fig. 3. Dimensionless profiles of the periodic waves ($\eta_* = \eta/H$), which are calculated by formulas (2.5) and (2.6) for $h_2/h_1 = 3$, $\rho_2/\rho_1 = 1.25$, $a = 0.3$, and $k_* = 3$ (a) or 33 (b): curves 1 and 2 refer to calculation by the approximation (2.6) for $\alpha = 0$ and 2/3, respectively.

and the peaks are sharper for positive values of μ.

3. Solitary Solutions of the Model Equation. We return back to Eq. (2.4) and integrate it once more over the variable ξ. (To satisfy the zeroth boundary conditions at infinities, we assume that the integration constant is zero.) Thus, we are led to the second-order nonlinear differential equation

$$(U^2 - g\delta A_\xi)\eta - \beta \frac{U^4}{g_+^-} A_\xi \frac{d^2\eta}{d\xi^2} - \zeta U^2\Big(1 + \frac{1}{2}A_\xi\Big)\eta^2$$

$$+\gamma\beta \frac{U^4}{g_+^-} A_\xi \frac{d^2\eta^2}{d\xi^2} - \frac{\delta}{2} U^2 A_\xi\Big[\Big(\frac{d\eta}{\partial\xi}\Big)^2 + \frac{7U^4}{(g_+^-)^2}\Big(\frac{d^2\eta}{d\xi^2}\Big)^2\Big] = 0. \tag{3.1}$$

We consider the case of sufficiently long perturbations ($kh_m < \varepsilon^{1/2}$). Then, we can consider that $A_\xi = 1$ in the second and third terms of Eq. (3.1) and omit the last two terms. Here, all the corrections that are ignored are not smaller than the third-order infinitesimals. If $\alpha = 2/3$, we finally obtain a very simple equation that takes into account the weak nonlinearity and long-wave dispersion of the perturbations:

$$(U^2 - g\delta)\eta - \frac{3}{2}\zeta U^2\eta^2 - \beta\frac{U^2}{g_+^-}\Big(U^2 - \frac{2}{3}g\delta\Big)\frac{d^2\eta}{d\xi^2} = 0. \tag{3.2}$$

The solutions of this equation can be expressed in terms of the Jacobi elliptic functions, i.e., they are cnoidal waves. In particular, for solitary perturbations we seek the solution in the form

$$\eta = \eta_a/\cosh^2\xi_s, \qquad \xi_s = \xi/L. \tag{3.3}$$

Here L is the characteristic longitudinal dimension of the wave. Then the second derivative $d^2\eta/d\xi^2$ is expressed through the square of the hyperbolic tangent:

$$\frac{d^2\eta}{d\xi^2} = \frac{2\eta}{L^2}(3\tanh^2\xi_s - 1). \tag{3.4}$$

We rewrite Eq. (3.2) with the use of the substitutions (3.3) and (3.4) in the form

$$U^2 = g\delta + \frac{3\zeta U^2}{2\cosh^2\xi_s}\eta + 2\frac{\beta}{g_+^-}\frac{U^2}{L^2}\Big(U^2 - \frac{2}{3}g\delta\Big)(3\tanh^2\xi_s - 1). \tag{3.5}$$

From this formula, for $\xi = 0$ ($\tanh\xi_s = 0$ and $\cosh\xi_s = 1$) we find the equality

$$U^2 = g\delta\Big[1 + \frac{3U^2}{2g\delta}\zeta\eta_a - \frac{2\beta U^2}{g_+^- L^2}\Big(\frac{U^2}{g\delta} - \frac{2}{3}\Big)\Big]. \tag{3.6}$$

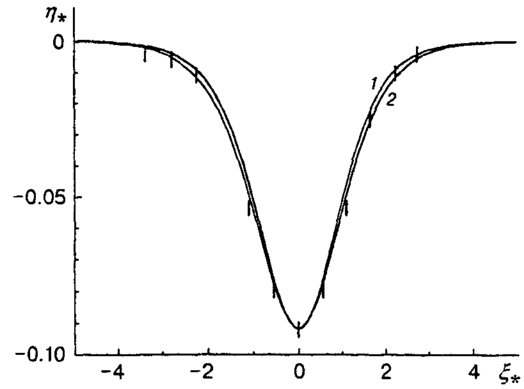

Fig. 4. Profile of the solitary wave ($\xi_* = \xi/H$) for $h_2/h_1 = 3$ and $\rho_2/\rho_1 = 1.25$: curve 1 refers to calculation by formulas (3.3) and (3.11) and curve 2 refers to calculation by formulas (3.3), (3.8), and (3.9); vertical sections refer to the experimental data [3] with account of the measurement error.

Similarly, for $\xi \to \infty$ ($\tanh \xi_s \to 1$ and $\cosh \xi_s \to \infty$) relation (3.5) yields

$$U^2 = g\delta\Big[1 + \frac{4\beta U^2}{g_+^- L^2}\Big(\frac{U^2}{g\delta} - \frac{2}{3}\Big)\Big]. \tag{3.7}$$

Adding (3.6) to (3.7) multiplied by 2, we obtain

$$U = c_0/\sqrt{1 - \eta_a^*}, \qquad \eta_a^* = \zeta\eta_a. \tag{3.8}$$

Substituting (3.8) into (3.7), we find

$$L = \frac{2h_1h_2(\rho_1 + \rho_2)}{\chi}\sqrt{\frac{1 + 2\eta_a^*}{3\eta_a^*(1 - \eta_a^*)}}. \tag{3.9}$$

The dependencies (3.8) and (3.9) are close to the respective equalities for the solitary solutions of the Korteweg–de Vries equation [6]

$$U_1 = c_0\Big(1 + \frac{\eta_a^*}{2}\Big) = U[1 + O(\varepsilon^2)], \qquad L_1 = 2\sqrt{\frac{h_1h_2(\rho_1h_1 + \rho_2h_2)}{3\chi\eta_a^*}}. \tag{3.10}$$

Note also that formulas (3.8) and (3.9) are in agreement with the characteristics of the solitary perturbations of the Boussinesq equation [7]

$$U_2 = c_0\sqrt{1 + \eta_a^*} = U[1 + O(\varepsilon^2)], \qquad L_2 = L_1\sqrt{1 + \eta_a^*}. \tag{3.11}$$

Thus, the soliton velocities in the Korteweg–de Vries and Boussinesq equations nearly coincide with the value for the wave found (for the same amplitudes), and some discrepancies between their lengths are due to the values of the ratios between the densities and depths of the layers. The previous studies of stationary waves have also shown that the solitary perturbation is determined by velocity U and length L_1 [8] or by Eqs. (3.11) [9].

The experimental form of the solitary perturbation [3] is compared with the analytical dependences in Fig. 4. One can see that all the results are in good agreement.

In the phase space of the solutions of Eq. (3.1), there are two equilibrium positions which correspond to the solutions η =const: $\eta_0 = 0$ and $\eta_1 = 2(1 - g\delta/U^2)/(3\zeta)$. We shall study the types of these singular points. Let $\eta = \eta_j + \eta'$, where $j = 0$ or $j = 1$, and η' be the infinitesimal. Then the linearized equation (3.1) can be rewritten in the form

$$\left(1-\frac{1}{U_\delta^2}-3\zeta\eta_j\right)\eta'=U_\beta^2(1-2\gamma\eta_j)\frac{d^2\eta'}{d\xi^2}-\alpha U_\beta^2\Big[\Big(\frac{1}{U_\delta^2}+\zeta\eta_j\Big)\frac{d^2\eta'}{d\xi^2}+U_\beta^2(1-2\gamma\eta_j)\frac{d^4\eta'}{d\xi^4}\Big].$$

Here $U_\delta^2=U^2/(g\delta)$ and $U_\beta^2=U^2\beta/g_+^-$. For the simplest model ($\alpha=0$), the point $\eta=\eta_1$ is the saddle for $U_\delta^2<1$ and the center for $1<U_\delta^2<4\gamma/(4\gamma-3\zeta)$; in contrast, the zeroth state of rest is the center for $U_\delta^2<1$ and the saddle for $U_\delta^2>1$. The term $d^4\eta'/d\xi^4$ can be ignored in the case of quite long perturbations. If $\alpha=2/3$ (second-order model), the zeroth equilibrium position is the center for $2/3<U_\delta^2<1$ and the saddle for $U_\delta^2>1$, and the point $\eta=\eta_1$ is the saddle for $U_\delta^2<1$ and the center for $1<U_\delta^2<(12\gamma-2\zeta)/(12\gamma-5\zeta)$. The upper boundary of the solitary-wave amplitude can also be found with the use of the latter inequality: $\eta_a<1/(6\gamma-\zeta)$.

4. Perturbation Transition from a Deep to a Shallow Fluid. We consider the propagation of a plane linear monochromatic wave with frequency ω in the direction of increase of the x coordinate: $\eta=a(x)\exp[i(\theta(x)-\omega t)]$. This is possible only if the layer depths (they vary smoothly) also depend only on the x coordinate. Consequently, Eqs. (1.12) become noticeably simpler:

$$\left(1+\frac{h_l}{g_+^-}A_\omega\omega^2\right)\omega^2\eta+\frac{c}{h_l}\frac{dh_l}{dx}i\omega\eta+(-1)^lA_\omega\left(gh_l\frac{\partial^2\eta}{\partial x^2}+\frac{h_l}{\rho_l}\frac{\partial^2p_i}{\partial x^2}\right)$$
$$+\frac{\alpha\beta h_l\omega^2}{g_+^-h_1h_2\chi}\left[(-1)^l\left(g\frac{\partial\eta}{\partial x}+\frac{1}{\rho_l}\frac{\partial p_i}{\partial x}\right)-\frac{c}{g_+^-}i\omega^3\eta\right]\left(\rho_1h_2^2\frac{dh_1}{dx}+\rho_2h_1^2\frac{dh_2}{dx}\right)=0. \qquad (4.1)$$

The expression $(-1)^l\langle u_l\rangle=c\eta/h_l$ is used to derive Eqs. (4.1). These equations are reduced to one equation for η, as in Sec. 2:

$$(1+\omega_*^2A_\omega)\omega^2\eta+g\delta A_\omega\frac{\partial^2\eta}{\partial x^2}$$
$$+\left[\frac{c}{h_1h_2\chi}i\omega\eta+\frac{\alpha\beta(\rho_1+\rho_2)\omega^2}{g_+^-\chi^2}\left(g_+^-\frac{\partial\eta}{\partial x}-\frac{c}{g_+^-}i\omega^3\eta\right)\right]\left(\rho_1h_2^2\frac{dh_1}{dx}+\rho_2h_1^2\frac{dh_2}{dx}\right)=0. \qquad (4.2)$$

Owing to the assumption, we have $\partial\eta/\partial x=(\eta/a)\,da/dx+i\eta\,d\theta/dx$ and

$$\frac{\partial^2\eta}{\partial x^2}=\frac{\eta}{a}\frac{d^2a}{dx^2}-\eta\left(\frac{d\theta}{dx}\right)^2+i\left(2\frac{\eta}{a}\frac{da}{dx}\frac{d\theta}{dx}+\eta\frac{d^2\theta}{dx^2}\right). \qquad (4.3)$$

Let the modulus of the derivative of function $\theta(x)$ be much larger than that of the derivative of the function $a(x)$. Then the first term in the right side of formula (4.3) can be omitted. These expressions for the derivatives enable us to rewrite Eq. (4.2) in the form of a system of equations (by dividing the equation into real and imaginary parts) for the functions $a(x)$ and $\theta(x)$ which determine the perturbation:

$$(1+\omega_*^2A_\omega)\omega^2a(x)=g\delta A_\omega\left(\frac{d\theta}{dx}\right)^2a(x)+\frac{\alpha\beta(\rho_1+\rho_2)}{\chi^2}\omega^2\frac{da}{dx}\left(\rho_1h_2^2\frac{dh_1}{dx}+\rho_2h_1^2\frac{dh_2}{dx}\right); \qquad (4.4)$$

$$g\delta A_\omega\left(\frac{d^2\theta}{dx^2}a(x)+2\frac{da}{dx}\frac{d\theta}{dx}\right)$$
$$+\left(\frac{c\omega}{h_1h_2\chi}(1-\alpha\omega_*^4)a(x)+\frac{\alpha\beta(\rho_1+\rho_2)}{\chi^2}\omega^2a(x)\frac{d\theta}{dx}\right)\left(\rho_1h_2^2\frac{dh_1}{dx}+\rho_2h_1^2\frac{dh_2}{dx}\right)=0. \qquad (4.5)$$

In the approximation under study, the second term on the right side of formula (4.4) can be ignored; then, $d\theta/dx$ corresponds to the wavenumber k. If $\rho_1=0$ and $h_1=$, from the dispersion relation we have the relation

$$\frac{dk}{dx}=-\frac{c\omega(1+2\alpha\omega_*^2)}{2gh_2^2(1+\alpha\omega_*^2)^2}\frac{dh_2}{dx}.$$

By substituting this expression and $d\theta/dx=k$ into Eq. (4.5), we obtain the following differential equation

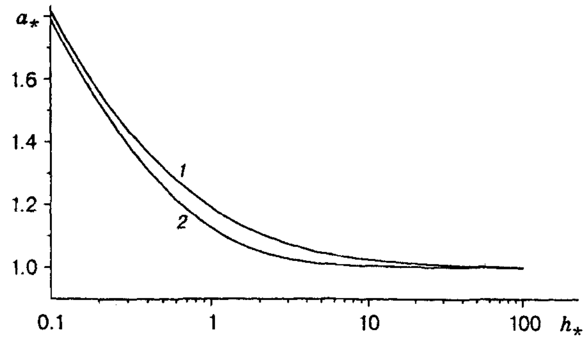

Fig. 5. Variation of the dimensionless wave amplitude during its smooth transition from deep to shallow layers: curves 1 and 2 refer to calculation by formula (4.8) for $\alpha = 0$ and 2/3, respectively.

with the separated variables a and h:

$$\frac{da_*}{a_*} = -\frac{dh_{2*}(1+2\alpha h_{2*})}{4h_{2*}(1+\alpha h_{2*})[1+h_{2*}(1+\alpha h_{2*})]}. \tag{4.6}$$

Here $a_* = a(x)/a_\infty$ and $h_{2*} = h_2 k_\infty = h_2\omega^2/g$, where a_∞ is the wave amplitude for an infinite depth of the lower fluid (the initial amplitude). The solution of Eq. (4.6) is a simple power-fractional relation

$$a_* = \left[1 + \frac{1}{h_{2*}\,(1+\alpha h_{2*})}\right]^{1/4}. \tag{4.7}$$

Formula (4.7) with $\alpha = 0$ was obtained in [10] for waves at the free surface of a homogeneous fluid.

The solutions of the second-order model can also be found as was done in [10]. We assume that $h_1 = h_2 = h$ and the system of linearized equations (1.1) and (1.10) is reduced to one equation for the liquid flow rate in the layer $q = \langle u_l \rangle h_l$:

$$\frac{\partial^2 q}{\partial t^2} - g_+^- h A_t \frac{\partial^2 q}{\partial x^2} - \frac{h}{g_+^-} A_t \frac{\partial^4 q}{\partial t^4} = 0.$$

If $q = Q(x)\exp(i[\theta(x) - \omega t])$, the real part of this equation is again in agreement with the dispersion relation and its imaginary part gives the conservation law $k(x)Q^2(x) = \text{const}$. Using any equation in (1.1), one can rewrite the latter in the form: $a^2(x)/k(x) = \text{const}$. Using the Padé approximation of the dispersion relation (1.5), we obtain the same power-fractional relation

$$a_* = \left[1 + \frac{1}{h_*(1+\alpha h_*)}\right]^{1/4}, \tag{4.8}$$

where $h_* = hk_\infty = h\omega^2/g_+^-$. The curves for various values of α are compared in Fig. 5.

Conclusions. The main results of the study are as follows:

1. A second-order model for two-dimensional weakly nonlinear waves of arbitrary length in a two-layer fluid with a gently sloping bottom and lid has been proposed. The approximation error for the phase velocity does not exceed 2%.

2. An evolution equation for progressive quasistationary perturbations which, among others, has stable solutions of the Stokes-wave type has been derived. In the limiting cases, the resulting dependence of the wave form on the frequency and the parameters of the system with horizontal boundaries gives known results

3. Sufficiently long perturbations can also be cnoidal waves. In particular, the velocity and profile of solitary perturbations were found to be in good agreement with the characteristics of not only the solitons of the Korteweg–de Vries and Boussinesq equations but also with the experimental data.

4. The problem of a smooth transition of a linear monoharmonic wave from deep to shallow layers has been studied. A comparison with the results obtained for perturbations of the free surface of a homogeneous fluid has been made.

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